Scalar phantom energy as a cosmological dynamical system

L. Arturo Ureña-López*

Instituto de Física de la Universidad de Guanajuato,
A.P. E-143, C.P. 37150, León, Guanajuato, México

(Dated: February 5, 2008)

Phantom energy can be visualized as a scalar field with a (non-canonical) negative kinetic energy term. We use the dynamical system formalism to study the attractor behavior of a cosmological model containing a phantom scalar field ϕ endowed with an exponential potential of the form $V(\phi) = V_0 \exp(-\lambda\kappa\phi)$, and a perfect fluid with constant equation of state γ; the latter can be of the phantom type too. As in the canonical case, three characteristic solutions can be identified. The scaling solution exists but is either unstable or of no physical interest. Thus, there are only two stable critical points which are of physical interest, corresponding to the perfect fluid and scalar field dominated solutions, respectively. The most interesting case arises for $0 > \gamma > -\lambda^2/3$, which allows the coexistence of the three solutions. The main features of each solution are discussed in turn.

I. INTRODUCTION

For many years, scalar fields have played the role of work-horses for many phenomenological models in modern Cosmology. And the main issue is to find which solutions of the field equations are physically important to explain the universe we live in.

The usual route was to get expertise in solving the coupled Einstein-Klein-Gordon (EKG) set of nonlinear differential equations, as it can be seen in the early literature on scalar fields. However, it was recently realized that for a exponential potential the EKG equations could be seen as a plane autonomous system of equations[1]. That is, the EKG equations could be seen as a dynamical system. This case became one more among the many examples of dynamical systems in Cosmology, see[2].

Even though the exponential potential is one of the most studied cases in the cosmology of scalar fields (see[3, 4, 5, 6] and references therein), the way it is handled in[1] allows a better understanding of the evolution of a universe with exponential potentials. The idea has been exploited in some papers that have used it successfully for diverse situations. For instance, in the cases of negative exponentials[7], an exponential potential in general Robertson-Walker spacetimes[8], and multi-exponential potentials[9, 10]; see also[11].

On the other hand, the idea of a scalar field with non-canonical kinetic energy has drawn the attention of cosmologists because such a scalar field can have, in principle, what is called a *phantom* equation of state[12, 13, 14, 15, 16, 17], for which $p/\rho < -1$, being p and ρ the pressure and the energy density of the phantom fluid, respectively. This new type of fluids violates the so-called dominant energy condition, which for a perfect fluid is written as $|p| \leq \rho$; for a thoroughly

discussion on this condition and the stability of vacuum see[14, 15]. Thus, the case of phantom scalar fields should be taken with care, as some even think that a cosmological constant may be easier to justify[13, 14, 15]. The presence of a kind of phantom matter may be supported by supernovae type Ia observations, as phantom matter seems to fit them better than a cosmological constant or a quintessence field [18, 19, 20]. Other alternatives, though, are a canonical scalar field climbing up its scalar potential[21, 22], quantum corrections on large scales[23, 24], and a non-canonical complex scalar field[25]; mechanisms that may result in a phantom-like equation of state.

Recently, the attractor behavior of phantom scalar field models, which have a negative kinetic term, was studied in[26] using the Hamilton-Jacobi formalism. Such study was restricted to cosmological models in which the phantom scalar field was the only matter present, using three different types of scalar potentials (power-law, exponential and cosine).

In the present paper, our aim is to perform a more complete study of the attractor properties of a phantom scalar field endowed with a positive exponential potential, using the dynamical system formalism developed in[1] for canonical scalar fields. This study will include a perfect fluid with a constant barotropic equation of state.

In this respect, it should be said that a similar analysis was performed in[27, 28], where a coupling between the phantom scalar field and a dust fluid was included. The main difference with that work is that we will not consider any coupling, and will not restrict the perfect fluid equation of state to the usual values; we will allow the possibility of super-stiff and phantom perfect fluids.

There are many other authors that have studied the cosmologies containing a phantom field, and have obtained explicit solutions together with careful stability analysis, see[26, 27, 28, 29, 30, 31, 32, 33, 34, 35] and the long list of references there in. We will comment on any similarity or difference that may appear with respect to

*Electronic address: lurena@fisica.ugto.mx

those papers.

A brief summary of the contents of this paper is as follows. In Sec II, we will review the mathematics needed for a cosmological model with a perfect fluid and a phantom scalar field. In Sec. III, the critical points of the system and their stability are presented. The discussion is focused in the three different cases that appear according to the values the perfect-fluid equation of state can take. Finally, Sec. IV is devoted to conclusions.

II.   MATHEMATICAL BACKGROUND

For simplicity, we shall restrict ourselves to the case of a homogeneous and isotropic universe with zero curvature, its metric given by the flat Robertson-Walker one[39]. In this universe, there is a perfect fluid with pressure p_γ and energy density ρ_γ related through a constant equation of state γ in the form $p_\gamma = (\gamma - 1)\rho_\gamma$. There is also a phantom scalar field ϕ minimally coupled to gravity, whose Lagrangian density reads $\mathcal{L} = (1/2)\partial^\mu\phi\partial_\mu\phi - V(\phi)$, where $V(\phi)$ is called the scalar potential. Therefore, the equations of motion are the known EKG equations

$$\dot{H} = -\frac{\kappa^2}{2}\left(\gamma\rho_\gamma - \dot{\phi}^2\right), \tag{1a}$$

$$\ddot{\phi} = -3H\dot{\phi} + \frac{dV}{d\phi}, \tag{1b}$$

$$\dot{\rho}_\gamma = -3H\gamma\rho_\gamma, \tag{1c}$$

together with the constraint (Friedmann) equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\left(\rho_\gamma - \frac{1}{2}\dot{\phi}^2 + V\right), \tag{2}$$

where dots mean derivative with respect to cosmic time, $\kappa^2 = 8\pi G$, H is the Hubble parameter, and $a(t)$ is the scale factor of the universe. The above equations are the same as obtained in the canonical case except for the minus sign in front of the scalar kinetic term $\dot{\phi}^2$.

We will study the case of a scalar exponential potential of the form $V(\phi) = V_0 \exp(-\lambda\kappa\phi)$, where V_0 and λ are free parameters. The similarity with the canonical case suggests that we should make use of the dimensionless variables[1]

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \tag{3}$$

and then Eqs. (1) can be written as a plane-autonomous system

$$x' = -3x + \frac{3}{2}\left[\gamma\left(1 + x^2 - y^2\right) - 2x^2\right]x - \sqrt{\frac{3}{2}}\lambda y^2, \tag{4a}$$

$$y' = \frac{3}{2}\left[\gamma\left(1 + x^2 - y^2\right) - 2x^2\right]y - \sqrt{\frac{3}{2}}\lambda xy, \tag{4b}$$

where now a prime denotes derivative with respect to the logarithm of the scale factor $N \equiv \ln a$. The constraint equation (2) now reads

$$1 = \frac{\kappa^2\rho_\gamma}{3H^2} - x^2 + y^2. \tag{5}$$

There are two scalar quantities we shall be interested in, which are the phantom density parameter Ω_ϕ and the effective phantom equation of state γ_ϕ, given by

$$\Omega_\phi = -x^2 + y^2, \quad \Omega_\phi\gamma_\phi = -2x^2. \tag{6}$$

We notice that the scalar equation of state γ_ϕ is always negative.

It is interesting to note that there will be three different regions in the phase plane, characterized by $\Omega_\phi < 0$, $0 \leq \Omega_\phi \leq 1$, and $\Omega_\phi > 1$, respectively. The boundary lines between the above regions are given by the hyperbola $y^2 - x^2 = 1$ and the straight lines $y = \pm x$, and are shown with dashed curves in the figures below.

As we will focus our attention in $\rho_\gamma > 0$, we see the Friedmann constraint (2) requires all realistic trajectories in the phase plane $\{x, y\}$ to comply with $\Omega_\phi \leq 1$. However, in opposition to the canonical case, there is no reason to constraint Ω_ϕ only to positive values. As we shall see later, the existence of perfect-fluid *supra-dominated* eras cannot be discarded.

III.   CRITICAL POINTS AND STABILITY

Eqs. (4) are written as an autonomous phase system of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, and its so-called critical points $\mathbf{x}_0$ are solutions of the system of equations $\mathbf{f}(\mathbf{x}_0) = 0$. To determine their stability we need to perform linear perturbations around the critical points in the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$, which results in the following equations of motion $\mathbf{u}' = \mathcal{M}\mathbf{u}$, where

$$\mathcal{M}_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}_0}. \tag{7}$$

A non-trivial critical point is called stable (unstable) whenever the eigenvalues m_i of $\mathcal{M}$ are such that $\mathrm{Re}(m_i) < 0$ ($\mathrm{Re}(m_i) > 0$). If neither of the aforementioned cases is accomplished, the critical point is called a saddle point[2].

We applied the above procedure to Eqs. (4). The critical points together with the stability analysis, and the corresponding values of Ω_ϕ and γ_ϕ are shown in Table I[40]. For completeness, the eigenvalues of matrix (7) for the critical points A, B, C and D are shown in Table II. For solution E, the eigenvalues are

$$m_\pm = -\frac{3}{4}|2 - \gamma|\left[\frac{(2 - \gamma)}{|2 - \gamma|} \pm \sqrt{1 - \frac{8\gamma\left(\lambda^2 + 3\gamma\right)}{\lambda^2\left(2 - \gamma\right)}}\right]. \tag{8}$$

TABLE I: Critical points for a phantom scalar field endowed with an exponential potential.

Label	ρ_γ	$\mathbf{x}_0$	Existence	Stability	Ω_ϕ	γ_ϕ
A	ρ_Λ	$(0,0)$	$\forall\lambda$	Saddle	0	Undefined
B	ρ_Λ	$\left(-\lambda/\sqrt{6},\sqrt{1+\lambda^2/6}\right)$	$\forall\lambda$	Stable	1	$-\lambda^2/3$
C	>0	$(0,0)$	$\forall\lambda,\forall\gamma$	Unstable for $\gamma>2$	0	Undefined
				Saddle for $2>\gamma>0$		
				Stable for $\gamma<0$		
D	>0	$\left(-\lambda/\sqrt{6},\sqrt{1+\lambda^2/6}\right)$	$\forall\lambda,\forall\gamma$	Stable for $\gamma>-\lambda^2/3$	1	$-\lambda^2/3$
				Saddle for $\gamma<-\lambda^2/3$		
E	>0	$\left(\sqrt{3/2}\gamma/\lambda,\sqrt{3\gamma(\gamma-2)/2\lambda^2}\right)$	$\forall\lambda,\gamma>2$	Saddle for $\gamma>2$	<0	γ
			$\forall\lambda,\gamma<0$	Saddle for $0>\gamma>-\lambda^2/3$	$-3\gamma/\lambda^2$	
				Stable for $\gamma<-\lambda^2/3$	>1	

TABLE II: Eigenvalues found for the critical points A, B, C and D shown in Table I, see Eq. (7).

Label	m_-	m_+
A	-3	0
B	$-3-\lambda^2/2$	$-\lambda^2$
C	$-3+3\gamma/2$	$3\gamma/2$
D	$-3-\lambda^2/2$	$-3\gamma-\lambda^2$

There are many similarities in the critical points with respect to the canonical case, so we briefly review the main solutions in the latter, see Table I in[1]. There are three solutions that are relevant for Cosmology, which are respectively dubbed the perfect fluid dominated solution, the scalar field dominated solution, and the scaling solution (for an interesting discussion on the existence of scaling solutions, see[31]).

The fluid dominated solution is in general unstable, except in the case of a cosmological constant. The scalar field dominated solution only exists for $\lambda^2 < 6$, in which case is the only late-time attractor and is inflationary for $\lambda^2 < 2$. Also, this solution is a saddle point when a cosmological constant is present.

For both cases $3\gamma < \lambda^2$ and $6 < \lambda^2$, the only stable solution is the scaling one, in which the scalar energy density remains proportional to that of the perfect fluid with $\Omega_\phi = 3\gamma/\lambda^2$. This last property has been widely used in some models of scalar field dark matter[36, 37][41].

As we shall see, the minus sign of the kinetic energy introduces relevant changes for the phantom case. There are three different situations we are about to revise now.

A. $\gamma = 0$

As it was found in[27, 28], in a universe with a cosmological constant $\rho_\gamma = \rho_\Lambda$ and a phantom scalar field there is only one stable critical point, which is the scalar field dominated solution B. This same as in the canonical case, except that now there is not restriction on the values that λ can take[27, 28]. As we mentioned before,

this also implies that an exponential potential will come to dominate over a cosmological constant too; this is to be expected since $\gamma_\phi < 0$.

1. Only phantom scalar field

It should be realized that the case in which no other matter except the phantom scalar field is present is *not* equivalent to the above case with $\gamma = 0$. In the former, Eqs. (4), and (5) can be fused into the single equation

$$x' = -\left(3x + \sqrt{\frac{3}{2}}\lambda\right)\left(1 + x^2\right). \tag{9}$$

Clearly, the only physically reasonable critical point is the same as point B[29][42], which is again stable. This time, however, point A cannot exist.

B. $\gamma > 0$

On the other hand, when a $(\gamma \neq 0)$ perfect fluid is present, things become more interesting. For $\gamma > 0$, the scaling solution E either does not exist or is not of physical interest, and the perfect fluid dominated solution C is in general unstable. Therefore, we conclude that the (phantom) scalar field dominated solutions B and D are unavoidable if $\gamma \geq 0$, which is the case for ordinary matter (radiation, dust, and even quintessence fields and a cosmological constant)[28]. The effective equation of state γ_ϕ is of the phantom type, and the scalar field climbs up its scalar potential. This is shown in Fig. 1 for $\gamma = 1$.

This is the first example in this work where we see *supra-dominance* of the perfect fluid over the phantom field, i.e. $\Omega_\gamma > 1$. Such a phenomenon, that passed unnoticed in previous published works[28, 29], also arises if the perfect fluid and the phantom field are coupled, as in[27].

Also in Fig. 1, we plotted trajectories that start with $\Omega_\phi = 0$ (at the boundary lines $y = \pm x$). We believe that
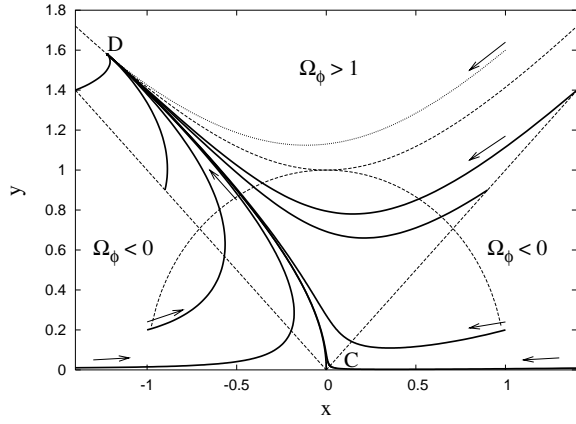
FIG. 1: The phase plane for $\gamma = 1$ and $\lambda = 3$. For $\gamma \geq 0$, the only late-time attractor is the stable critical point D. Notice that, in opposition to the canonical scalar field, there is not restriction on the value of λ. We also show the phantom-divide (dashed) circumference, and some trajectories where the perfect fluid is *supra-dominant*, see text for details. For completeness, we show a representative trajectory with $\Omega_\phi > 1$ (dotted curves), which never enters the physically interesting region $\Omega_\phi \leq 1$. For the case in which there is not a perfect fluid present, all trajectories lie on the hyperbola $y^2 - x^2 = 1$ and finish at the critical point D.

all of such trajectories are indeed part of longer trajectories which may have started in the negative-Ω_ϕ regions.

It is also worth noticing that trajectories can start with $\Omega_\phi < 0$, but always end up with $0 \leq \Omega_\phi \leq 1$, but not vice-versa; this is because there are not stable critical points with $\Omega_\phi < 0$ (in the case of coupling between matter and the phantom field, there are stable critical points in which the perfect fluid matter is supra-dominant, see[27]). Moreover, the parabola $y^2 - x^2 = 1$ is not traversable at all, as trajectories with $\Omega_\phi > 1$ never enters into the region $0 \leq \Omega_\phi \leq 1$ in order to reach the stable critical point C.

To better visualize the evolution of the universe along the trajectories in Fig. 1, it is convenient to define an effective equation of state of the universe $\gamma_{\rm eff}$, which for our case reads

$$\gamma_{\rm eff} \equiv \frac{\gamma \rho_\gamma + \gamma_\phi \rho_\phi}{\rho_\gamma + \rho_\phi} = \gamma \left[1 - y^2 - \left(\frac{2 - \gamma}{\gamma} \right) x^2 \right] . \quad (10)$$

The value $\gamma_{\rm eff} = 0$ defines, in general, an ellipse (hyperbola) in the xy plane if $0 < \gamma < 2$ ($\gamma > 2$). That is to say, this ellipse (and this hyperbola) marks the phantom divide of the universe in the phase plane: $\gamma_{\rm eff} > 0$ ($\gamma_{\rm eff} < 0$) inside (outside) the ellipse (and the hyperbola).

For $\gamma = 1$, the phantom divide is a unitary circumference, as shown in Fig. 1. A curious point is that some trajectories cross over the phantom divide twice. Even though the perfect fluid can be supra-dominant, the effective equation of the universe rarely coincides with that of the perfect fluid.
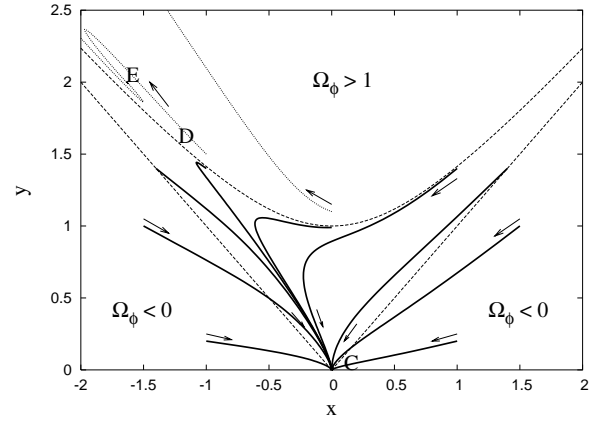

FIG. 2: The phase plane for $\gamma = -4$ and $\lambda = 3$. For this case, the only late-time attractor is the stable critical point C, i.e., the (phantom) perfect fluid dominates over the scalar field, even if a stage of supra-dominance is allowed initially. For completeness, we show numerical solutions of the case $\Omega_\phi > 1$ (dotted curves), for which point E is an attractor stable solution, which is in agreement with the stability analysis in Eq. (8).

C. $\gamma < 0$

There is an important change in the analysis above if we allow for $\gamma < 0$, that is, if the perfect fluid is also phantom. For $\gamma < -\lambda^2/3$, D is a saddle point (the scalar field dominated solution cannot be sustained), and point E is again of no physical interest even if it is a stable node. The only possibility is then the (phantom) perfect fluid dominated solution C (the perfect fluid is more phantom than the scalar field). The phase plane for an instance of this case is shown in Fig. 2.

There is still one extra case we have to care about. For $0 > \gamma > -\lambda^2/3$, both points C and D exist and are *stable*. Moreover, point E is also physically allowed, but this time is a saddle point. The numerical experience, as shown in Fig. 3, indicates that some of the trajectories pass nearby point E before finishing at one of the late-attractors C and D. How close a trajectory is to point E depends on the initial conditions $\mathbf{x}_0$.

We also see that all trajectories with initial perfect fluid supra-dominance never reach point D, and then always end up at critical point C. As for the $\Omega > 1$ case, we see that all trajectories end up at point D, but again without crossing over the $\Omega_\phi \leq 1$ region.

As a final note, it can be seen that the phantom-divide curve for $\gamma < 0$ is a hyperbola, see Eq. (10); and then $\gamma_{\rm eff}$ is positive only well within the ($\Omega_\phi > 1$)-region of the phase plane.

IV. CONCLUSIONS

The analysis of the EKG equations from the point of view of dynamical systems is very fruitful, as the appear-
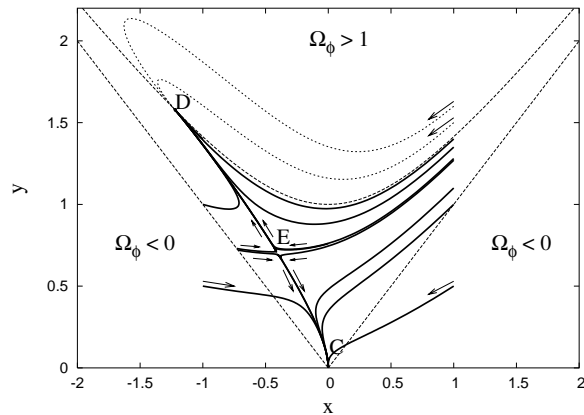
FIG. 3: The phase plane for $\gamma = -1$ and $\lambda = 3$. For this case, there are *two* late-time attractors, which are the critical points C and D; though the saddle point E is also present. Which of the stable critical points is reached at the end, depends on the initial values $\mathbf{x}_0$. Notice that point D is also a late-time attractor for $\Omega_\phi > 1$ (dotted curves).

ance of critical points helps a lot to reveal the role scalar fields play in many cosmological models.

We have studied the case of a phantom scalar field endowed with an exponential potential, and found that there are in general only two stable solutions which are of physical interest. For matter with $\gamma \geq 0$, it is confirmed that the scalar field dominated solution is the only late-time attractor.

There is for this case the same problem as in the canonical scalar field[1]. In order to have a cosmological evolution with standard radiation and matter dominated eras, the scalar field contribution should be almost negligible at early times. If the phantom scalar field were initially present at the time of inflation, the inflaton field could not avoid a phantom era unless $\Omega_\phi \ll 1$ at the beginning of inflation. This is because the critical point A is a saddle point, as we have numerically confirmed, but trajectories *move* away from it only linearly with N.

As in the canonical case, once the radiation and matter dominated eras appear, the phantom scalar field density parameter grows like $\Omega_\phi \sim a^{3\gamma}$, which also means that the scalar energy density remains almost constant until it reaches its late-time attractor solution. Hence, in order to avoid a disturbing phantom dominated universe, we need the same constraint that appears in the standard case, that is, the initial scalar density parameter should be such that $\Omega_{\phi,i} < \rho_{\gamma,0}/\rho_{\gamma,i}$, in order not to become the dominant component at time t_0.

The novel case that appears for a phantom equation of state γ is the coexistence of three critical points if $0 > \gamma > -\lambda^2/3$; a case that can also happen when matters are coupled[27]. Allowing for more than one phantom type of matter leads to a non-trivial evolution; we learn in this case that the ultimate phantom regime will depend on the kind of phantoms present. For instance, it is seen from Fig. 3 that the most phantom matter may not be the dominant one at the end. This result may be relevant in models in which more than one type of phantom matter coexist together. In[34], the authors report a lower bound in the effective equation of state in the case of many phantom scalars with an $\mathcal{O}(N)$ symmetry. This suggests that the classical solutions of phantom scalar fields may have hidden properties worth investigating.

Another result is that *supra-dominance* of the perfect fluid $\Omega_\gamma > 1$ cannot be discarded in the case of a phantom scalar field, because the scalar field contribution Ω_ϕ cannot be, in principle, bounded from below. If phantom scalar fields were permitted in Nature, there would have to be a mechanism that can prevent Ω_ϕ to become too negative.

Strangely enough, for $0 < \gamma < 2$ the effective equation of state of the universe γ_{eff} can be negative even in the case of supra-dominance; moreover, it can change from negative to positive values, and vice-versa, along some trajectories.

The model presented in this work has the same properties that appears in any single field-phantom model in which $\gamma_\phi < 0$: big rip singularity and no crossing of the $\gamma_\phi = 0$ barrier, see[30, 38] for more details.

In studying such properties, the dynamical system formalism is not only useful but also easy. Apart from helping to decide which solutions of the EKG system are of physical interest, it also simplifies the numerical solutions.

The exponential case fits well into the dynamical system formalism, as has been extensively shown in the literature, but the formalism can be easily adapted for other scalar potentials. We expect to report on them elsewhere.

Acknowledgments

[1] E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. **D57**, 4686 (1998), gr-qc/9711068.

[2] J. Wainwright and G. F. R. Ellis, eds., *Dynamical systems in Cosmology* (Cambridge UP, 1997).

[3] V. Sahni (2004), astro-ph/0403324.

[4] N. Ohta, Int. J. Mod. Phys. **A20**, 1 (2005), hep-th/0411230.

[5] I. P. Neupane, Class. Quant. Grav. **21**, 4383 (2004), hep-

th/0311071.

[6] C.-M. Chen, P.-M. Ho, I. P. Neupane, and J. E. Wang, JHEP **07**, 017 (2003), hep-th/0304177.

[7] I. P. C. Heard and D. Wands, Class. Quant. Grav. **19**, 5435 (2002), gr-qc/0206085.

[8] R. J. van den Hoogen, A. A. Coley, and D. Wands, Class. Quant. Grav. **16**, 1843 (1999), gr-qc/9901014.

[9] Z. K. Guo, Y.-S. Piao, and Y.-Z. Zhang, Phys. Lett. **B568**, 1 (2003), hep-th/0304048.

[10] A. Collinucci, M. Nielsen, and T. Van Riet, Class. Quant. Grav. **22**, 1269 (2005), hep-th/0407047.

[11] P. G. Vieira, Class. Quant. Grav. **21**, 2421 (2004), hep-th/0311173.

[12] R. R. Caldwell, Phys. Lett. **B545**, 23 (2002), astro-ph/9908168.

[13] G. W. Gibbons (2003), hep-th/0302199.

[14] J. M. Cline, S.-y. Jeon, and G. D. Moore, Phys. Rev. **D70**, 043543 (2004), hep-ph/0311312.

[15] S. M. Carroll, M. Hoffman, and M. Trodden, Phys. Rev. **D68**, 023509 (2003), astro-ph/0301273.

[16] S. Nojiri and S. D. Odintsov, Phys. Rev. **D72**, 023003 (2005), hep-th/0505215.

[17] I. P. Neupane and D. L. Wiltshire (2005), hep-th/0504135.

[18] U. Alam, V. Sahni, and A. A. Starobinsky, JCAP **0406**, 008 (2004), astro-ph/0403687.

[19] H. K. Jassal, J. S. Bagla, and T. Padmanabhan, Mon. Not. Roy. Astron. Soc. **356**, L11 (2005), astro-ph/0404378.

[20] H. K. Jassal, J. S. Bagla, and T. Padmanabhan (2005), astro-ph/0506748.

[21] C. Csaki, N. Kaloper, and J. Terning (2005), astro-ph/0507148.

[22] M. Sahlen, A. R. Liddle, and D. Parkinson (2005), astro-ph/0506696.

[23] V. K. Onemli and R. P. Woodard, Phys. Rev. **D70**, 107301 (2004), gr-qc/0406098.

[24] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. **19**, 4607 (2002), gr-qc/0204065.

[25] H. Wei, R.-G. Cai, and D.-F. Zeng, Class. Quant. Grav. **22**, 3189 (2005), hep-th/0501160.

[26] Z.-K. Guo, Y.-S. Piao, and Y.-Z. Zhang, Phys. Lett. **B594**, 247 (2004), astro-ph/0404225.

[27] B. Gumjudpai, T. Naskar, M. Sami, and S. Tsujikawa, JCAP **0506**, 007 (2005), hep-th/0502191.

[28] J.-g. Hao and X.-z. Li, Phys. Rev. **D67**, 107303 (2003), gr-qc/0302100.

[29] E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. **D70**, 043539 (2004), hep-th/0405034.

[30] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. **D71**, 063004 (2005), hep-th/0501025.

[31] F. Piazza and S. Tsujikawa, JCAP **0407**, 004 (2004), hep-th/0405054.

[32] P. Singh, M. Sami, and N. Dadhich, Phys. Rev. **D68**, 023522 (2003), hep-th/0305110.

[33] J.-G. Hao and X.-z. Li, Phys. Rev. **D70**, 043529 (2004), astro-ph/0309746.

[34] X.-z. Li and J.-g. Hao, Phys. Rev. **D69**, 107303 (2004), hep-th/0303093.

[35] X.-z. Li, Y.-b. Zhao, and C.-b. Sun (2005), astro-ph/0508019.

[36] T. Matos and L. A. Ureña Lopez, Int. J. Mod. Phys. **D13**, 2287 (2004), astro-ph/0406194.

[37] V. Sahni and L.-M. Wang, Phys. Rev. **D62**, 103517 (2000), astro-ph/9910097.

[38] S. Nojiri and S. D. Odintsov, Phys. Lett. **B595**, 1 (2004), hep-th/0405078.

[39] We will be using a metric signature $(-, +, +, +)$, and units in which $c = 1$

[40] There are other two critical points $\mathbf{x}_0 = (\pm i, 0)$, which are the counterparts of the kinetic dominated solutions in the canonical case. But, they are obviously not of physical interest this time, see also Table I in[27].

[41] It should be noticed that the *scaling* solution E in Table I is very similar to the so called *tracker* attractor solution for phantom fields[33]; though the latter does not exist for an exponential potential.

[42] This is at variance with the results in[26], where the authors claim that there is indeed the same restriction as in the canonical case, namely $\lambda^2 < 6$, for the existence of this solution. In our study, we have found no evidence for such restriction, see also Fig. 1 below.